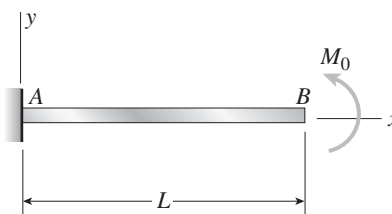


Differential Equations of the Deflection Curve

The beams described in the problems for Section 9.4 have constant flexural rigidity EI . Also, the origin of coordinates is at the left-hand end of each beam.

Problem 9.4-1 Derive the equation of the deflection curve for a cantilever beam AB when a couple M_0 acts counterclockwise at the free end (see figure). Also, determine the deflection δ_B and slope θ_B at the free end. Use the third-order differential equation of the deflection curve (the shear-force equation).



Solution 9.4-1 Cantilever beam (couple M_0)

SHEAR-FORCE EQUATION (EQ. 9-12 b).

$$EIv''' = V = 0$$

$$EIv'' = C_1$$

B.C. 1 $M = M_0$ $EIv'' = M = M_0 = C_1$

$$EIv' = C_1x + C_2 = M_0x + C_2$$

B.C. 2 $v'(0) = 0$ $\therefore C_2 = 0$

$$EIv = \frac{M_0x^2}{2} + C_3$$

B.C. 3 $v(0) = 0$ $\therefore C_3 = 0$

$$v = \frac{M_0x^2}{2EI} \quad \leftarrow$$

$$v' = \frac{M_0x}{EI}$$

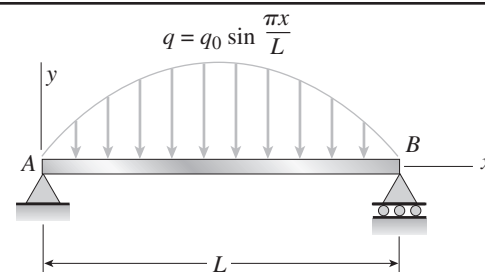
$$\delta_B = v(L) = \frac{M_0L^2}{2EI} \text{ (upward)} \quad \leftarrow$$

$$\theta_B = v'(L) = \frac{M_0L}{EI} \text{ (counterclockwise)} \quad \leftarrow$$

(These results agree with Case 6, Table G-1.)

Problem 9.4-2 A simple beam AB is subjected to a distributed load of intensity $q = q_0 \sin \pi x/L$, where q_0 is the maximum intensity of the load (see figure).

Derive the equation of the deflection curve, and then determine the deflection δ_{\max} at the midpoint of the beam. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-2 Simple beam (sine load)

LOAD EQUATION (EQ. 9-12 c).

$$EIv'''' = -q = -q_0 \sin \frac{\pi x}{L}$$

$$EIv''' = q_0 \left(\frac{L}{\pi} \right) \cos \frac{\pi x}{L} + C_1$$

$$EIv'' = q_0 \left(\frac{L}{\pi} \right)^2 \sin \frac{\pi x}{L} + C_1x + C_2$$

B.C. 1 $EIv'' = M$ $EIv''(0) = 0$ $\therefore C_2 = 0$

B.C. 2 $EIv''(L) = 0$ $\therefore C_1 = 0$

$$EIv' = -q_0 \left(\frac{L}{\pi} \right)^3 \cos \frac{\pi x}{L} + C_3$$

$$EIv = -q_0 \left(\frac{L}{\pi} \right)^4 \sin \frac{\pi x}{L} + C_3x + C_4$$

B.C. 3 $v(0) = 0$ $\therefore C_4 = 0$

B.C. 4 $v(L) = 0$ $\therefore C_3 = 0$

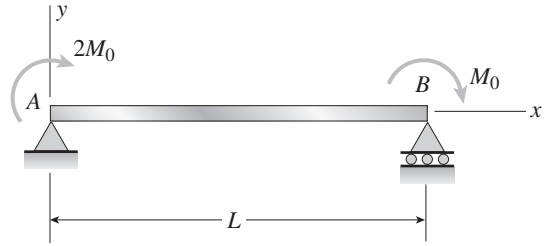
$$v = -\frac{q_0L^4}{\pi^4EI} \sin \frac{\pi x}{L} \quad \leftarrow$$

$$\delta_{\max} = -v\left(\frac{L}{2}\right) = \frac{q_0L^4}{\pi^4EI} \quad \leftarrow$$

(These results agree with Case 13, Table G-2.)

Problem 9.4-3 The simple beam AB shown in the figure has moments $2M_0$ and M_0 acting at the ends.

Derive the equation of the deflection curve, and then determine the maximum deflection δ_{\max} . Use the third-order differential equation of the deflection curve (the shear-force equation).



Solution 9.4-3 Simple beam with two couples

Reaction at support A: $R_A = \frac{3M_0}{L}$ (downward)

Shear force in beam: $V = -R_A = -\frac{3M_0}{L}$

SHEAR-FORCE EQUATION (EQ. 9-12 b)

$$EIv''' = V = -\frac{3M_0}{L}$$

$$EIv'' = -\frac{3M_0x}{L} + C_1$$

B.C. 1 $EIv'' = M$ $EIv''(0) = 2M_0 \quad \therefore C_1 = 2M_0$

$$EIv' = -\frac{3M_0x^2}{2L} + 2M_0x + C_2$$

$$EIv = -\frac{M_0x^3}{2L} + M_0x^2 + C_2x + C_3$$

B.C. 2 $v(0) = 0 \quad \therefore C_3 = 0$

B.C. 3 $v(L) = 0 \quad \therefore C_2 = -\frac{M_0L}{2}$

$$v = -\frac{M_0x}{2LEI}(L^2 - 2Lx + x^2) = -\frac{M_0x}{2LEI}(L-x)^2 \leftarrow$$

$$v' = -\frac{M_0}{2LEI}(L-x)(L-3x)$$

MAXIMUM DEFLECTION

Set $v' = 0$ and solve for x :

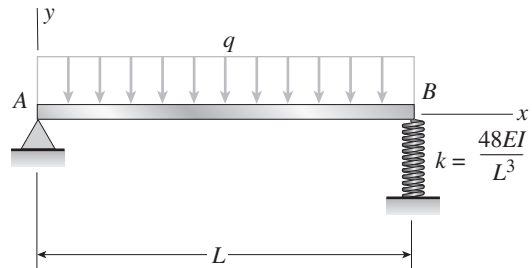
$$x_1 = L \text{ and } x_2 = \frac{L}{3}$$

Maximum deflection occurs at $x_2 = \frac{L}{3}$.

$$\delta_{\max} = -v\left(\frac{L}{3}\right) = \frac{2M_0L^2}{27EI} \text{ (downward)} \leftarrow$$

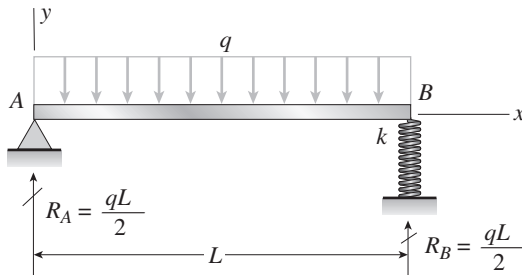
Problem 9.4-4 A simple beam with a uniform load is pin supported at one end and spring supported at the other. The spring has stiffness $k = 48EI/L^3$.

Derive the equation of the deflection curve by starting with the third-order differential equation (the shear-force equation). Also, determine the angle of rotation θ_A at support A.



Solution 9.4-4 Beam with a spring support

REACTIONS



DEFLECTIONS AT END B

$$k = \frac{48EI}{L^3} \quad \delta_B = \frac{R_B}{k} = \frac{qL}{2k} = \frac{qL^4}{96EI}$$

SHEAR-FORCE EQUATION (EQ. 9-12 b)

$$V = R_A - qx = \frac{q}{2}(L - 2x)$$

$$EIv''' = V = \frac{q}{2}(L - 2x)$$

$$EIv'' = \frac{q}{2}(Lx - x^2) + C_1$$

$$\text{B.C. 1 } EIv'' = M \quad EIv''(0) = 0 \quad \therefore C_1 = 0$$

$$EIv' = \frac{q}{2}\left(\frac{Lx^2}{2} - \frac{x^3}{3}\right) + C_2$$

$$EIv = \frac{q}{2}\left(\frac{Lx^3}{6} - \frac{x^4}{12}\right) + C_2x + C_3$$

$$\text{B.C. 2 } v(0) = 0 \quad \therefore C_3 = 0$$

$$\text{B.C. 3 } v(L) = -\delta_B = -\frac{qL^4}{96EI}$$

$$\therefore C_2 = -\frac{5qL^3}{96}$$

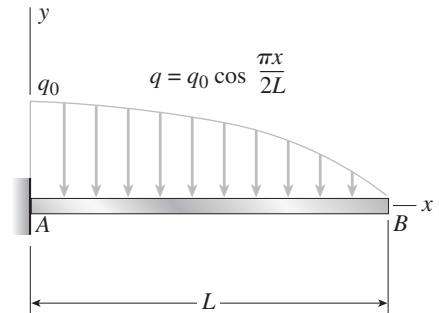
$$v = -\frac{qx}{96EI}(5L^3 - 8Lx^2 + 4x^3) \quad \leftarrow$$

$$v' = -\frac{q}{96EI}(5L^3 - 24Lx^2 + 16x^3)$$

$$\theta_A = -v'(0) = \frac{5qL^3}{96EI} \quad (\text{clockwise}) \quad \leftarrow$$

Problem 9.4-5 The distributed load acting on a cantilever beam AB has an intensity q given by the expression $q_0 \cos \pi x / 2L$, where q_0 is the maximum intensity of the load (see figure).

Derive the equation of the deflection curve, and then determine the deflection δ_B at the free end. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-5 Cantilever beam (cosine load)

LOAD EQUATION (EQ. 9-12 c)

$$EIv'''' = -q = -q_0 \cos \frac{\pi x}{2L}$$

$$EIv''' = -q_0 \left(\frac{2L}{\pi}\right) \sin \frac{\pi x}{2L} + C_1$$

$$\text{B.C. 1 } EIv''' = V \quad EIv'''(L) = 0 \quad \therefore C_1 = \frac{2q_0L}{\pi}$$

$$EIv'' = q_0 \left(\frac{2L}{\pi}\right)^2 \cos \frac{\pi x}{2L} + \frac{2q_0Lx}{\pi} + C_2$$

$$\text{B.C. 2 } EIv'' = M \quad EIv''(L) = 0 \quad \therefore C_2 = -\frac{2q_0L^2}{\pi}$$

$$EIv' = q_0 \left(\frac{2L}{\pi}\right)^3 \sin \frac{\pi x}{2L} + \frac{q_0Lx^2}{\pi} - \frac{2q_0L^2x}{\pi} + C_3$$

$$\text{B.C. 3 } v'(0) = 0 \quad \therefore C_3 = 0$$

$$EIv = -q_0 \left(\frac{2L}{\pi}\right)^4 \cos \frac{\pi x}{2L} + \frac{q_0Lx^3}{3\pi} - \frac{q_0L^2x^2}{\pi} + C_4$$

$$\text{B.C. 4 } v(0) = 0 \quad \therefore C_4 = \frac{16q_0L^4}{\pi^4}$$

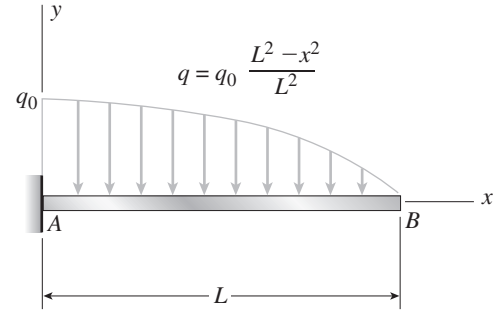
$$v = -\frac{q_0L}{3\pi^4EI} \left(48L^3 \cos \frac{\pi x}{2L} - 48L^3 + 3\pi^3 Lx^2 - \pi^3 x^3 \right) \quad \leftarrow$$

$$\delta_B = -v(L) = \frac{2q_0L^4}{3\pi^4EI} (\pi^3 - 24) \quad \leftarrow$$

(These results agree with Case 10, Table G-1.)

Problem 9.4-6 A cantilever beam AB is subjected to a parabolically varying load of intensity $q = q_0(L^2 - x^2)/L^2$, where q_0 is the maximum intensity of the load (see figure).

Derive the equation of the deflection curve, and then determine the deflection δ_B and angle of rotation θ_B at the free end. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-6 Cantilever beam (parabolic load)

LOAD EQUATION (EQ. 9-12 c)

$$EIv'''' = -q = -\frac{q_0}{L^2}(L^2 - x^2)$$

$$EIv''' = -\frac{q_0}{L^2}\left(L^2x - \frac{x^3}{3}\right) + C_1$$

B.C. 1 $EIv''' = V$ $EIv'''(L) = 0 \quad \therefore C_1 = \frac{2q_0L}{3}$

$$EIv'' = -\frac{q_0}{L^2}\left(\frac{L^2x^2}{2} - \frac{x^4}{12}\right) + \frac{2q_0L}{3}x + C_2$$

B.C. 2 $EIv'' = M$ $EIv''(L) = 0 \quad \therefore C_2 = -\frac{q_0L^2}{4}$

$$EIv' = -\frac{q_0}{L^2}\left(\frac{L^2x^3}{6} - \frac{x^5}{60}\right) + \frac{q_0Lx^2}{3} - \frac{q_0L^2x}{4} + C_3$$

B.C. 3 $v'(0) = 0 \quad \therefore C_3 = 0$

$$EIv = -\frac{q_0}{L^2}\left(\frac{L^2x^4}{24} - \frac{x^6}{360}\right) + \frac{q_0Lx^3}{9} - \frac{q_0L^2x^2}{8} + C_4$$

B.C. 4 $v(0) = 0 \quad \therefore C_4 = 0$

$$v = -\frac{q_0x^2}{360L^2EI}(45L^4 - 40L^3x + 15L^2x^2 - x^4) \leftarrow$$

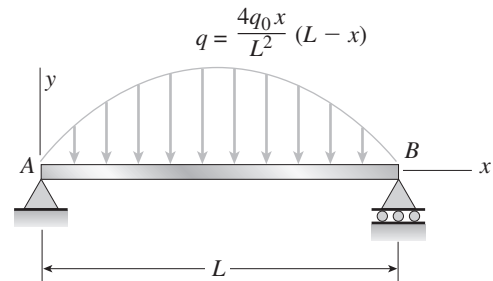
$$\delta_B = -v(L) = \frac{19q_0L^4}{360EI} \leftarrow$$

$$v' = -\frac{q_0x}{60L^2EI}(15L^4 - 20L^3x + 10L^2x^2 - x^4)$$

$$\theta_B = -v'(L) = \frac{q_0L^3}{15EI} \leftarrow$$

Problem 9.4-7 A beam on simple supports is subjected to a parabolically distributed load of intensity $q = 4q_0x(L - x)/L^2$, where q_0 is the maximum intensity of the load (see figure).

Derive the equation of the deflection curve, and then determine the maximum deflection δ_{\max} . Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-7 Single beam (parabolic load)

LOAD EQUATION (EQ. 9-12 c)

$$EIv'''' = -q = -\frac{4q_0x}{L^2}(L - x) = -\frac{4q_0}{L^2}(Lx - x^2)$$

$$EIv''' = -\frac{2q_0}{3L^2}(3Lx^2 - 2x^3) + C_1$$

$$EIv'' = -\frac{q_0}{3L^2}(2Lx^3 - x^4) + C_1x + C_2$$

B.C. 1 $EIv'' = M$ $EIv''(0) = 0 \quad \therefore C_2 = 0$

B.C. 2 $EIv''(L) = 0 \quad \therefore C_1 = \frac{q_0L}{3}$

$$EIv' = -\frac{q_0}{30L^2}(-5L^3x^2 + 5Lx^4 - 2x^5) + C_3$$

B.C. 3 (Symmetry) $v'\left(\frac{L}{2}\right) = 0 \quad \therefore C_3 = -\frac{q_0L^3}{30}$

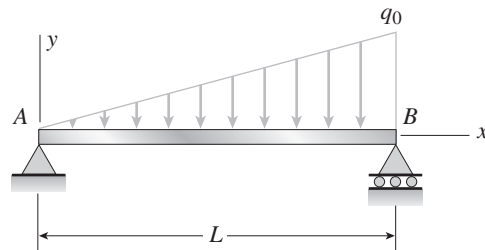
$$EIv = -\frac{q_0}{30L^2}\left(L^5x - \frac{5L^3x^3}{3} + Lx^5 - \frac{x^6}{3}\right) + C_4$$

B.C. 4 $v(0) = 0 \quad \therefore C_4 = 0$

$$v = -\frac{q_0x}{90L^2EI}(3L^5 - 5L^3x^2 + 3Lx^4 - x^5) \leftarrow$$

$$\delta_{\max} = -v\left(\frac{L}{2}\right) = \frac{61q_0L^4}{5760EI} \leftarrow$$

Problem 9.4-8 Derive the equation of the deflection curve for a simple beam AB carrying a triangularly distributed load of maximum intensity q_0 (see figure). Also, determine the maximum deflection δ_{\max} of the beam. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-8 Simple beam (triangular load)

LOAD EQUATION (EQ. 9-12 c)

$$EIv'''' = -q = -\frac{q_0 x}{L} \quad EIv'''' = -\frac{q_0 x^2}{2L} + C_1$$

$$EIv''' = -\frac{q_0 x^3}{6L} + C_1 x + C_2$$

B.C. 1 $EIv'' = M \quad EIv''(0) = 0 \quad \therefore C_2 = 0$

B.C. 2 $EIv''(L) = 0 \quad \therefore C_1 = \frac{q_0 L}{6}$

$$EIv' = -\frac{q_0 x^4}{24L} + \frac{q_0 L x^2}{12} + C_3$$

$$EIv = -\frac{q_0 x^5}{120L} + \frac{q_0 L x^3}{36} + C_3 x + C_4$$

B.C. 3 $v(0) = 0 \quad \therefore C_4 = 0$

B.C. 4 $v(L) = 0 \quad \therefore C_3 = -\frac{7q_0 L^3}{360}$

$$v = -\frac{q_0 x}{360 LEI} (7L^4 - 10L^2 x^2 + 3x^4) \quad \leftarrow$$

$$v' = -\frac{q_0}{360 LEI} (7L^4 - 30L^2 x^2 + 15x^4)$$

MAXIMUM DEFLECTION

Set $v' = 0$ and solve for x :

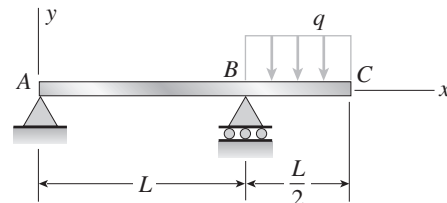
$$x_1^2 = L^2 \left(1 - \sqrt{\frac{8}{15}} \right) \quad x_1 = 0.51933L$$

$$\delta_{\max} = -v(x_1) = \frac{q_0 L^4}{225 EI} \left(\frac{5}{3} + \frac{2}{3} - \sqrt{\frac{8}{15}} \right)^{1/2} \quad \leftarrow$$

$$= 0.006522 \frac{q_0 L^4}{EI} \quad \leftarrow$$

(These results agree with Case 11, Table G-2.)

Problem 9.4-9 Derive the equations of the deflection curve for an overhanging beam ABC subjected to a uniform load of intensity q acting on the overhang (see figure). Also, obtain formulas for the deflection δ_C and angle of rotation θ_C at the end of the overhang. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-9 Beam with an overhang

LOAD EQUATION (EQ. 9-12 c)

$$EIv'''' = -q = 0 \quad (0 \leq x \leq L)$$

$$EIv'''' = C_1 \quad (0 \leq x \leq L)$$

$$EIv''' = C_1 x + C_2 \quad (0 \leq x \leq L)$$

B.C. 1 $EIv'' = M \quad EIv''(0) = 0 \quad \therefore C_2 = 0$

$$EIv'''' = -q \quad \left(L \leq x \leq \frac{3L}{2} \right)$$

$$EIv''' = -qx + C_3 \quad \left(L \leq x \leq \frac{3L}{2} \right)$$

B.C. 2 $EIv''' = V \quad EIv''' \left(\frac{3L}{2} \right) = 0 \quad \therefore C_3 = \frac{3qL}{2}$

$$EIv'' = -\frac{qx^2}{2} + \frac{3qLx}{2} + C_4 \quad \left(L \leq x \leq \frac{3L}{2} \right)$$

B.C. 3 $EIv'' = M \quad EIv'' \left(\frac{3L}{2} \right) = 0 \quad \therefore C_4 = -\frac{9qL^2}{8}$

B.C. 4 $EI(v'')_{\text{Left}} = EI(v'')_{\text{Right}} \quad \text{at } x = L$

$$C_1 L = -\frac{qL^2}{2} + \frac{3qL^2}{2} - \frac{9qL^2}{8} \quad \therefore C_1 = -\frac{qL}{8}$$

$$EIv' = -\frac{qLx^2}{16} + C_5 \quad (0 \leq x \leq L)$$

$$EIv' = -\frac{qx^3}{6} + \frac{3qLx^2}{4} - \frac{9qL^2x}{8} + C_6$$

$$\left(L \leq x \leq \frac{3L}{2} \right)$$

$$\text{B.C. 5 } (v')_{\text{Left}} = (v')_{\text{Right}} \text{ at } x = L$$

$$\therefore C_6 = C_5 + \frac{23qL^3}{48}$$

$$EIv = -\frac{qLx^3}{48} + C_5x + C_7 \quad (0 \leq x \leq L)$$

$$\text{B.C. 6 } v(0) = 0 \quad \therefore C_7 = 0$$

$$\text{B.C. 7 } v(L) = 0 \text{ for } 0 \leq x \leq L \quad \therefore C_5 = \frac{qL^3}{48}$$

$$\text{From Eq.(a): } C_6 = \frac{qL^3}{2}$$

$$EIv = -\frac{qx^4}{24} + \frac{3qLx^3}{12} - \frac{9qL^2x^2}{16} + \frac{qL^3x}{2} + C_8 \quad \left(L \leq x \leq \frac{3L}{2}\right)$$

$$\text{B.C. 8 } v(L) = 0 \text{ for } L \leq x \leq \frac{3L}{2} \quad \therefore C_8 = -\frac{7qL^4}{48}$$

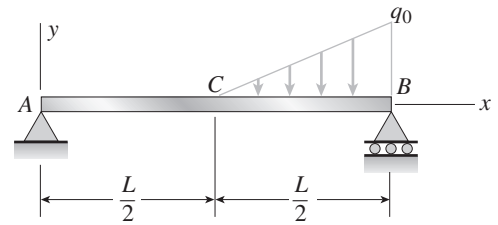
$$v = -\frac{qLx}{48EI}(L^2 - x^2) \quad (0 \leq x \leq L) \quad \leftarrow$$

$$v = -\frac{q(L-x)}{48EI}(7L^3 - 17L^2x + 10Lx^2 - 2x^3) \quad \leftarrow \quad \left(L \leq x \leq \frac{3L}{2}\right)$$

$$\delta_C = -v\left(\frac{3L}{2}\right) = \frac{11qL^4}{384EI} \quad \leftarrow$$

$$\theta_C = -v'\left(\frac{3L}{2}\right) = \frac{qL^3}{16EI} \quad \leftarrow$$

Problem 9.4-10 Derive the equations of the deflection curve for a simple beam AB supporting a triangularly distributed load of maximum intensity q_0 acting on the right-hand half of the beam (see figure). Also, determine the angles of rotation θ_A and θ_B at the ends and the deflection δ_C at the midpoint. Use the fourth-order differential equation of the deflection curve (the load equation).



Solution 9.4-10 Simple beam (triangular load)

LOAD EQUATION (EQ. 9-12 c)

$$\text{Left-hand half (part AC): } 0 \leq x \leq \frac{L}{2}$$

$$\text{Right-hand half (part CB): } \frac{L}{2} \leq x \leq L$$

PART AC $q = 0$

$$EIv'''' = -q = 0 \quad EIv'''' = C_1$$

$$EIv'' = C_1x + C_2 \quad EIv' = C_1\left(\frac{x^2}{2}\right) + C_2x + C_3$$

$$EIv = C_1\left(\frac{x^3}{6}\right) + C_2\left(\frac{x^2}{2}\right) + C_3x + C_4$$

PART CB $q = \frac{q_0}{L}(2x - L)$

$$EIv'''' = -q = \frac{q_0}{L}(L - 2x)$$

$$EIv''' = \frac{q_0}{L}(Lx - x^2) + C_5$$

$$EIv'' = \frac{q_0}{L}\left(\frac{Lx^2}{2} - \frac{x^3}{3}\right) + C_5x + C_6$$

$$EIv' = \frac{q_0}{L}\left(\frac{Lx^3}{6} - \frac{x^4}{12}\right) + C_5\left(\frac{x^2}{2}\right) + C_6x + C_7$$

$$EIv = \frac{q_0}{L}\left(\frac{Lx^4}{24} - \frac{x^5}{60}\right) + C_5\left(\frac{x^3}{6}\right) + C_6\left(\frac{x^2}{2}\right) + C_7x + C_8$$

BOUNDARY CONDITIONS

$$\text{B.C. 1 } EIv''' = V \quad EI(v''')_{AC} = EI(v''')_{BC} \text{ at } x = \frac{L}{2}$$

$$C_1 - C_5 = \frac{q_0L}{4} \quad (1)$$

$$\text{B.C. 2 } EIv'' = M \quad EIv''(0) = 0 \quad C_2 = 0 \quad (2)$$

$$\text{B.C. 3 } EIv'(L) = 0 \quad C_5L + C_6 = -\frac{q_0L^2}{6} \quad (3)$$

$$\text{B.C. 4 } (EIv'')_{AC} = (EIv'')_{CB} \text{ for } x = \frac{L}{2} \quad C_1L - C_5L - 2C_6 = \frac{q_0L^2}{6} \quad (4)$$

$$\text{B.C. 5 } (v')_{AC} = (v')_{CB} \quad \text{for } x = \frac{L}{2}$$

$$C_1 L^2 + 8C_3 - C_5 L^2 - 4C_6 L - 8C_7 = \frac{q_0 L^3}{8} \quad (5)$$

$$\text{B.C. 6 } v(0) = 0 \quad C_4 = 0$$

$$\text{B.C. 7 } v(L) = 0$$

$$C_5 L^3 + 3C_6 L^2 + 6C_7 L + 6C_8 = -\frac{3q_0 L^4}{20} \quad (7)$$

$$\text{B.C. 8 } (v)_{AC} = (v)_{CB} \quad \text{for } x = \frac{L}{2}$$

$$C_1 L^3 + 24C_3 L - C_5 L^3 - 6C_6 L^2 - 24C_7 L - 48C_8 = \frac{q_0 L^4}{10} \quad (8)$$

SOLVE EQS. (1) THROUGH (8):

$$C_1 = \frac{q_0 L}{24} \quad C_2 = 0 \quad C_3 = -\frac{37q_0 L^3}{5760}$$

$$C_4 = 0 \quad C_5 = -\frac{5q_0 L}{24} \quad C_6 = \frac{q_0 L^2}{24}$$

$$C_7 = -\frac{67q_0 L^3}{5760} \quad C_8 = \frac{q_0 L^4}{1920}$$

Substitute constants into equations for v and v' .

DEFLECTION CURVE FOR PART AC $\left(0 \leq x \leq \frac{L}{2}\right)$

$$v = -\frac{q_0 L x}{5760 EI} (37L^2 - 40x^2) \quad \leftarrow$$

$$(6) \quad v' = -\frac{q_0 L}{5760 EI} (37L^2 - 120x^2)$$

$$\theta_A = -v'(0) = \frac{37q_0 L^3}{5760 EI} \quad \leftarrow$$

$$\delta_C = -v\left(\frac{L}{2}\right) = \frac{3q_0 L^4}{1280 EI} \quad \leftarrow$$

DEFLECTION CURVE FOR PART CB $\left(\frac{L}{2} \leq x \leq L\right)$

$$(8) \quad v = -\frac{q_0}{5760 LEI} [L^2 x (37L^2 - 40x^2) + 3(2x - L)^5] \quad \leftarrow$$

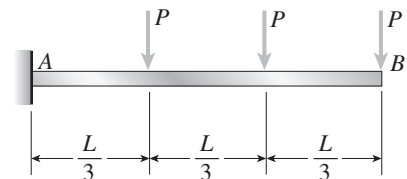
$$v' = -\frac{q_0}{5760 LEI} [L^2 (37L^2 - 120x^2) + 30(2x - L)^4]$$

$$\theta_B = v'(L) = \frac{53q_0 L^3}{5760 EI} \quad \leftarrow$$

Method of Superposition

The problems for Section 9.5 are to be solved by the method of superposition. All beams have constant flexural rigidity EI .

Problem 9.5-1 A cantilever beam AB carries three equally spaced concentrated loads, as shown in the figure. Obtain formulas for the angle of rotation θ_B and deflection δ_B at the free end of the beam.



Solution 9.5-1 Cantilever beam with 3 loads

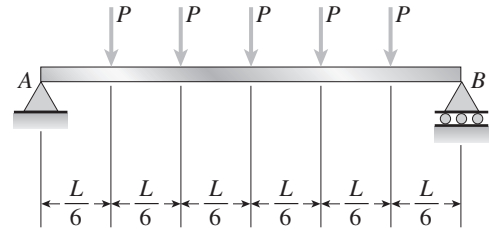
Table G-1, Cases 4 and 5

$$\theta_B = \frac{P\left(\frac{L}{3}\right)^2}{2EI} + \frac{P\left(\frac{2L}{3}\right)^2}{2EI} + \frac{PL^2}{2EI} = \frac{7PL^2}{9EI} \quad \leftarrow$$

$$\begin{aligned} \delta_B &= \frac{P\left(\frac{L}{3}\right)^2}{6EI} \left(3L - \frac{L}{3}\right) + \frac{P\left(\frac{2L}{3}\right)^2}{6EI} \left(3L - \frac{2L}{3}\right) + \frac{PL^3}{3EI} \\ &= \frac{5PL^3}{9EI} \quad \leftarrow \end{aligned}$$

Problem 9.5-2 A simple beam AB supports five equally spaced loads P (see figure).

- (a) Determine the deflection δ_1 at the midpoint of the beam.
 (b) If the same total load ($5P$) is distributed as a uniform load on the beam, what is the deflection δ_2 at the midpoint?
 (c) Calculate the ratio of δ_1 to δ_2 .



Solution 9.5-2 Simple beam with 5 loads

(a) Table G-2, Cases 4 and 6

$$\begin{aligned} \delta_1 &= \frac{P\left(\frac{L}{6}\right)}{24EI} \left[3L^2 - 4\left(\frac{L}{6}\right)^2 \right] + \\ &\quad \frac{P\left(\frac{L}{3}\right)}{24EI} \left[3L^2 - 4\left(\frac{L}{3}\right)^2 \right] + \frac{PL^3}{48EI} \\ &= \frac{11PL^3}{144EI} \quad \leftarrow \end{aligned}$$

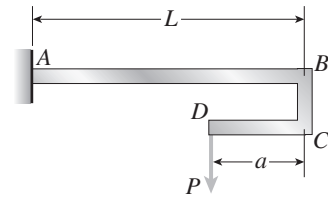
(b) Table G-2, Case 1 $qL = 5P$

$$\delta_2 = \frac{5qL^4}{384EI} = \frac{25PL^4}{384EI} \quad \leftarrow$$

$$(c) \frac{\delta_1}{\delta_2} = \frac{11}{144} \left(\frac{384}{25} \right) = \frac{88}{75} = 1.173 \quad \leftarrow$$

Problem 9.5-3 The cantilever beam AB shown in the figure has an extension BCD attached to its free end. A force P acts at the end of the extension.

- (a) Find the ratio a/L so that the vertical deflection of point B will be zero.
 (b) Find the ratio a/L so that the angle of rotation at point B will be zero.



Solution 9.5-3 Cantilever beam with extension

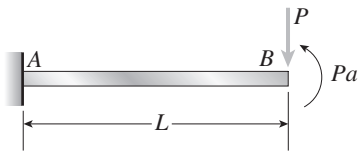


Table G-1, Cases 4 and 6

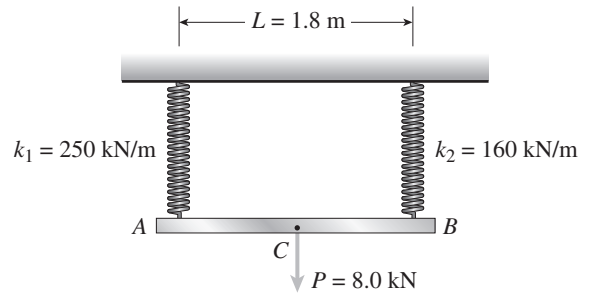
$$(a) \delta_B = \frac{PL^3}{3EI} - \frac{PaL^2}{2EI} = 0 \quad \frac{a}{L} = \frac{2}{3} \quad \leftarrow$$

$$(b) \theta_B = \frac{PL^2}{2EI} - \frac{PaL}{EI} = 0 \quad \frac{a}{L} = \frac{1}{2} \quad \leftarrow$$

Problem 9.5-4 Beam ACB hangs from two springs, as shown in the figure. The springs have stiffnesses k_1 and k_2 and the beam has flexural rigidity EI .

What is the downward displacement of point C , which is at the midpoint of the beam, when the load P is applied?

Data for the structure are as follows: $P = 8.0$ kN, $L = 1.8$ m, $EI = 216$ kN·m², $k_1 = 250$ kN/m, and $k_2 = 160$ kN/m.



Solution 9.5-4 Beam hanging from springs

$$P = 8.0 \text{ kN} \quad L = 1.8 \text{ m}$$

$$EI = 216 \text{ kN} \cdot \text{m}^2 \quad k_1 = 250 \text{ kN/m}$$

$$k_2 = 160 \text{ kN/m}$$

Stretch of springs:

$$\delta_A = \frac{P/2}{k_1} \quad \delta_B = \frac{P/2}{k_2}$$

Table G-2, Case 4

$$\delta_C = \frac{PL^3}{48EI} + \frac{1}{2} \left(\frac{P/2}{k_1} + \frac{P/2}{k_2} \right)$$

$$= \frac{PL^3}{48EI} + \frac{P}{4} \left(\frac{1}{k_1} + \frac{1}{k_2} \right) \quad \leftarrow$$

Substitute numerical values:

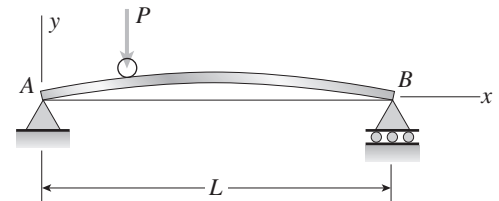
$$\delta_C = \frac{(8.0 \text{ kN})(1.8 \text{ m})^3}{48(216 \text{ kN} \cdot \text{m}^2)}$$

$$+ \frac{8.0 \text{ kN}}{4} \left(\frac{1}{250 \text{ kN/m}} + \frac{1}{160 \text{ kN/m}} \right) \quad \leftarrow$$

$$= 4.5 \text{ mm} + 20.5 \text{ mm}$$

$$= 25 \text{ mm} \quad \leftarrow$$

Problem 9.5-5 What must be the equation $y = f(x)$ of the axis of the slightly curved beam AB (see figure) *before* the load is applied in order that the load P , moving along the bar, always stays at the same level?



Solution 9.5-5 Slightly curved beam

Let x = distance to load P

δ = downward deflection at load P

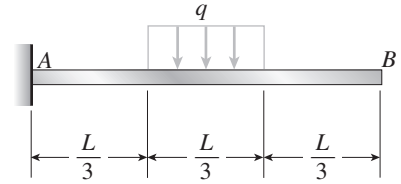
Table G-2, Case 5:

$$\delta = \frac{P(L-x)x}{6LEI} [L^2 - (L-x)^2 - x^2] = \frac{Px^2(L-x)^2}{3LEI}$$

Initial upward displacement of the beam must equal δ .

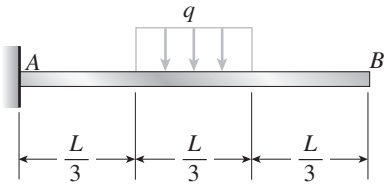
$$\therefore y = \frac{Px^2(L-x)^2}{3LEI} \quad \leftarrow$$

Problem 9.5-6 Determine the angle of rotation θ_B and deflection δ_B at the free end of a cantilever beam AB having a uniform load of intensity q acting over the middle third of its length (see figure).

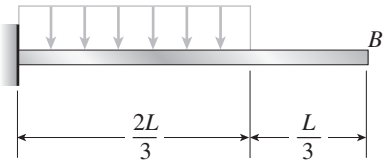


Solution 9.5-6 Cantilever beam (partial uniform load)

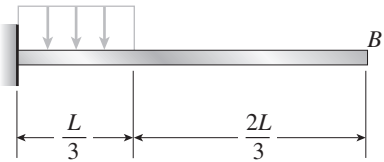
q = intensity of uniform load
Original load on the beam:



Load No. 1:



Load No. 2:



SUPERPOSITION: Original load = Load No. 1 minus Load No. 2

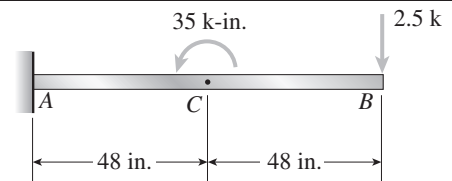
Table G-1, Case 2

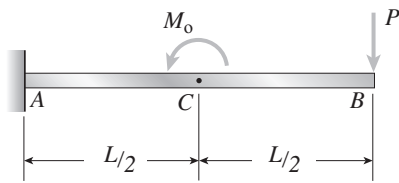
$$\theta_B = \frac{q}{6EI} \left(\frac{2L}{3}\right)^3 - \frac{q}{6EI} \left(\frac{L}{3}\right)^3 = \frac{7qL^3}{162EI} \quad \leftarrow$$

$$\delta_B = \frac{q}{24EI} \left(\frac{2L}{3}\right)^3 \left(4L - \frac{2L}{3}\right) - \frac{q}{24EI} \left(\frac{L}{3}\right)^3 \left(4L - \frac{L}{3}\right)$$

$$= \frac{23qL^4}{648EI} \quad \leftarrow$$

Problem 9.5-7 The cantilever beam ACB shown in the figure has flexural rigidity $EI = 2.1 \times 10^6$ k-in.² Calculate the downward deflections δ_C and δ_B at points C and B , respectively, due to the simultaneous action of the moment of 35 k-in. applied at point C and the concentrated load of 2.5 k applied at the free end B .



Solution 9.5-7 Cantilever beam (two loads)

$$EI = 2.1 \times 10^6 \text{ k-in.}^2$$

$$M_0 = 35 \text{ k-in.}$$

$$P = 2.5 \text{ k}$$

$$L = 96 \text{ in.}$$

Table G-1, Cases 4.6, and 7

$$\begin{aligned} \delta_C &= -\frac{M_0(L/2)^2}{2EI} + \frac{P(L/2)^2}{6EI} \left(3L - \frac{L}{2}\right) \\ &= -\frac{M_0L^2}{8EI} + \frac{5PL^3}{48EI} \quad (+ = \text{downward deflection}) \end{aligned}$$

$$\begin{aligned} \delta_B &= -\frac{M_0(L/2)}{2EI} \left(2L - \frac{L}{2}\right) + \frac{PL^3}{3EI} \\ &= -\frac{3M_0L^2}{8EI} + \frac{PL^3}{3EI} \quad (+ = \text{downward deflection}) \end{aligned}$$

SUBSTITUTE NUMERICAL VALUES:

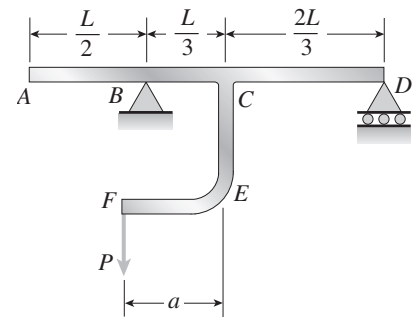
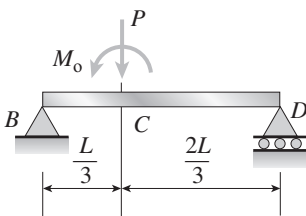
$$\begin{aligned} \delta_C &= -0.01920 \text{ in.} + 0.10971 \text{ in.} \\ &= 0.0905 \text{ in.} \quad \leftarrow \end{aligned}$$

$$\begin{aligned} \delta_B &= -0.05760 \text{ in.} + 0.35109 \text{ in.} \\ &= 0.293 \text{ in.} \quad \leftarrow \end{aligned}$$

Problem 9.5-8 A beam $ABCD$ consisting of a simple span BD and an overhang AB is loaded by a force P acting at the end of the bracket CEF (see figure).

(a) Determine the deflection δ_A at the end of the overhang.

(b) Under what conditions is this deflection upward? Under what conditions is it downward?

**Solution 9.5-8 Beam with bracket and overhang**

Consider part BD of the beam.

$$M_0 = Pa$$

Table G-2, Cases 5 and 9

$$\begin{aligned} \theta_B &= \frac{P(L/3)(2L/3)(5L/3)}{6LEI} \\ &\quad + \frac{Pa}{6LEI} \left[6\left(\frac{L^2}{3}\right) - 3\left(\frac{L^2}{9}\right) - 2L^2 \right] \\ &= \frac{PL}{162EI} (10L - 9a) \quad (+ = \text{clockwise angle}) \end{aligned}$$

(a) DEFLECTION AT THE END OF THE OVERHANG

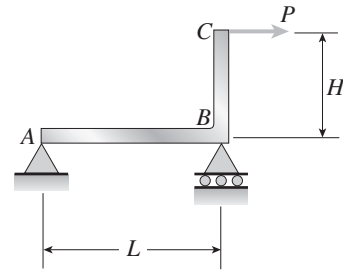
$$\begin{aligned} \delta_A &= \theta_B \left(\frac{L}{2}\right) = \frac{PL^2}{324EI} (10L - 9a) \quad \leftarrow \\ &\quad (+ = \text{upward deflection}) \end{aligned}$$

(b) Deflection is upward when $\frac{a}{L} < \frac{10}{9}$ and downward when $\frac{a}{L} > \frac{10}{9}$ \leftarrow

Problem 9.5-9 A horizontal load P acts at end C of the bracket ABC shown in the figure.

- (a) Determine the deflection δ_C of point C .
- (b) Determine the maximum upward deflection δ_{\max} of member AB .

Note: Assume that the flexural rigidity EI is constant throughout the frame. Also, disregard the effects of axial deformations and consider only the effects of bending due to the load P .



Solution 9.5-9 Bracket ABC

BEAM AB

$$M_0 = PH$$

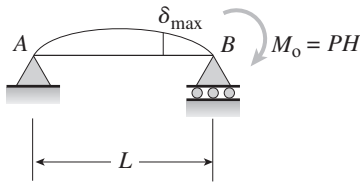


Table G-2, Case 7: $\theta_B = \frac{M_0 L}{3EI} = \frac{PHL}{3EI}$

(a) ARM BC Table G-1, Case 4

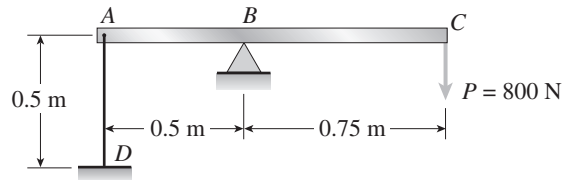
$$\begin{aligned} \delta_C &= \frac{PH^3}{3EI} + \theta_B H = \frac{PH^3}{3EI} + \frac{PH^2 L}{3EI} \\ &= \frac{PH^2}{3EI} (L + H) \quad \leftarrow \end{aligned}$$

(b) MAXIMUM DEFLECTION OF BEAM AB

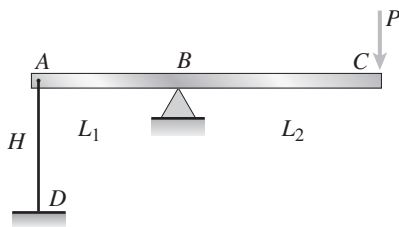
Table G-2, Case 7: $\delta_{\max} = \frac{M_0 L^2}{9\sqrt{3}EI} = \frac{PHL^2}{9\sqrt{3}EI} \quad \leftarrow$

Problem 9.5-10 A beam ABC having flexural rigidity $EI = 75 \text{ kN} \cdot \text{m}^2$ is loaded by a force $P = 800 \text{ N}$ at end C and tied down at end A by a wire having axial rigidity $EA = 900 \text{ kN}$ (see figure).

What is the deflection at point C when the load P is applied?



Solution 9.5-10 Beam tied down by a wire



- $EI = 75 \text{ kN} \cdot \text{m}^2$
- $P = 800 \text{ N}$
- $EA = 900 \text{ kN}$
- $H = 0.5 \text{ m} \quad L_1 = 0.5 \text{ m}$
- $L_2 = 0.75 \text{ m}$

CONSIDER BC AS A CANTILEVER BEAM

Table G-1, Case 4: $\delta'_C = \frac{PL_2^3}{3EI}$

CONSIDER AB AS A SIMPLE BEAM

$$M_0 = PL_2$$

Table G-2, Case 7: $\theta'_B = \frac{M_0 L_1}{3EI} = \frac{PL_1 L_2}{3EI}$

CONSIDER THE STRETCHING OF WIRE AD

$$\delta'_A = (\text{Force in } AD) \left(\frac{H}{EA} \right) = \left(\frac{PL_2}{L_1} \right) \left(\frac{H}{EA} \right) = \frac{PL_2 H}{EAL_1}$$

DEFLECTION δ_C OF POINT C

$$\begin{aligned} \delta_C &= \delta'_C + \theta'_B (L_2) + \delta'_A \left(\frac{L_2}{L_1} \right) \\ &= \frac{PL_2^3}{3EI} + \frac{PL_1 L_2^2}{3EI} + \frac{PL_2^2 H}{EAL_1^2} \quad \leftarrow \end{aligned}$$

SUBSTITUTE NUMERICAL VALUES:

$$\delta_C = 1.50 \text{ mm} + 1.00 \text{ mm} + 1.00 \text{ mm} = 3.50 \text{ mm} \quad \leftarrow$$